数理解析研究所講究録 第2128巻 2019年 18-32

MODELING AND INTEGRATION OF THE RISE AND FALL OF CAPILLARY ACTION BY POISSON.

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Abstract. We discuss the mathematical theory of deduction of the capillary action by Laplace, Gauss, Poisson. These share the common concept of attraction and repulsive force on continuum, which is realized with two constants. The former two deduces to the equations of the capillary surface, and the latter, Poisson [10] confirms the formulae, in another model and analytical problems. In the latter half, we introduce Poisson's applications of Legendre's elastic functions to the capillary model owing to the theory and table by Legendre in 1825-6ish [7], which Poisson gives up the self made theories based on the same elliptic functions including tables.

Mathematics Subject Classification 2010 : 76B40, 76D45, 01Axx, 35Qxx, 74AXX.
Key words : Capirary action, surface tension, mathematical history, mathematical physics, fluid statics, elliptic function by Legendre, Legendre's tables.

1. Poisson's paper of capillarity

1.1. Poisson's comments on Gauss [1].

Poisson [10] commented in the preface about Gauss [1]:

- Gauss' success is due to the merit of his $\prec$ *characteristic* $\succ$
- even Gauss uses the same method as the given physics by Laplace.
- Gauss calculates by the condition only the same density and incompressibility

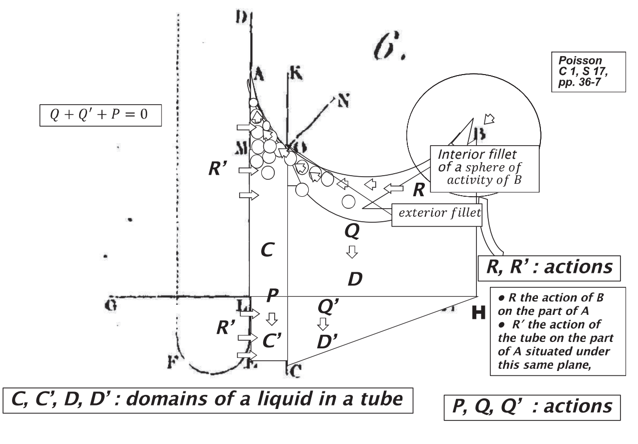

fig.1 the rise in the neighborhood of water surface and wall.

Date: 2018/01/19.

1.2. **Proof by Poisson that the rise in the neighborhood of water surface and wall is due to the abrupt variation of density.**

§ 14. Posed thus, call A the liquid contained in a vertical cylinder which has its base on the plane GH and which the generatrix is the straight DL tangent to the wall of the tube, and B the liquid situated around this cylinder and under GH. It goes along one which precedes that the vertical action of the tube and of B on A will independent of the inferior surface of the tube, which the vertical section is represented with $EC'F$, so that we will be capable to replace this surface with a horizontal plane. If we designate then with R the action of B on the part of A situated on this plane, and R' the action of the tube on the part of A situated under this same plane, and if we suppose that the primary force is exercises in the direction of the gravity, and the second in the contrary direction,

$$(7)_3 \qquad 2R' - R = \Delta, \tag{1}$$

for the equilibrium of A. It will rest now to formula the expressions of R and R'. Consequently, were ds an element infinitely small of contour of a ; with the two extremities of ds, trace the planes perpendicular to its direction which is cut along with a vertical passing with the center of the curvature of this contour ; let separate the segment of A composed between these two planes, and fillets infinitely thin with the plane vertical parallel to ds ; and were u the distance from one of these fillets at the plane vertical passing through ds. We will be capable to explain its basis with $(1 - ku)duds$, in supposing this fillet composed in the sphere of activity of B, neglecting the power of u superior to the primary, and designating with k, a constant coefficient which will depend on the curvature of the contour of a, at the point which responds to ds'. The basis of an exterior fillet, belonging to B, which responds to another element ds' of the contour, will be at the same time $(1 + k'u')ds'du'$; u' being the insensible distance from this second fillet to the surface of A, and k' this one which turns k at the point corresponding to ds'. From here, we will conclude without difficulty

$$R = \rho^2 \int\int\int\int\int\int \varphi(r)\frac{z+z'}{r}(1-ku)(1+k'u')dzdz'dudu'dsds',$$

in putting

$$r^2 = x^2 + (u+u')^2 + (z+z')^2.$$

§ 15. (The determination of R and R'.)

designating with $\varphi(r)$ the same function with preceding (no. 2), with x the projection of the arc composed between ds and ds' on the prolongation of ds and with z and z' the perpendicular fallen from a point of A and of a point B on the plane GH, so that r were the distance of a point to the other. At the degree of approximation where we are stayed in all this one which proceeds, we will turn to reduce to the unit the factors $1 - ku$ and $1 + k'u'$. We will be capable next of extending from zero to the infinite, the integrals in respect to u, u', z, z', and integral in respect to x from $x = -\infty$ to $x = +\infty$, namely only from $x = 0$ to $x = \infty$, in doubling the result. In putting

$$2\rho^2 \int\int\int\int\int \varphi(r)\frac{z+z'}{r}dzdz'dudu'dx \equiv q,$$

and taking the five integrals from zero to the infinity, we will have then $R = \int qds$.

This last integral will turn to all the points of the contour of a ; and as q won't turn to extend from a point to another, it is followed that if we call c the entire length of this contour, we will have simply $R = cq$. If we designate with $\varphi'(r)$ the mutual attraction of the material of the tube and of that of liquid, relative to the distance r and related at the unit of the volume, ρ', and ρ being the densities of the two materials, and if we represent with q' this one which q turns, when we put $\rho\rho'\varphi'(r)$ instead of $\rho^2\varphi(r)$, we will find similarly $R' = cq'$; by means of this equation (1) will be turn into

$$(8)_3 \qquad \Delta = (2q' - q)c. \tag{2}$$

§ 16. The quintuplicate integral which q represents is reduced easily to a simple integral. In putting at first zx, $z'x$, ux, $u'x$, xdz, xdz', instead of z, z', u and u', and of their primary differentials, the limits zero and the infinity won't changes ; it will result $q = 2X\rho^2 \int_0^\infty r^4\varphi(r)dr$, in putting, to abridge

$$X \equiv \int\int\int\int \frac{(z+z')dzdz'dudu'}{\Big[1+(u+u')^2+(z+z')^2\Big]^2},$$

The integration relative to the one of these five variables, to z' for example, is effectuated immediately ; and by reason of the limits $z'=0$ and $z'=\infty$, we have

$$X = \frac{1}{4}\int\int\int \frac{dzdudu'}{\Big[1+(u+u')^2+z^2\Big]^2}. \tag{3}$$

I put $z = y\sqrt{1+(u+u')^2}$, $\quad dz = dy\sqrt{1+(u+u')^2}$, this one gives [1]

$$X = \frac{1}{4}\int\int \frac{dudu'}{\Big[1+(u+u')^2\Big]^{\frac{3}{2}}}\int_0^\infty \frac{dy}{\Big(1+y^2\Big)^2}\ ;$$

[2] and as this integral relative to y is equal to $\frac{\pi}{4}$, we conclude then

$$X = \frac{\pi}{16}\int\int \frac{dudu'}{\Big[1+(u+u')^2\Big]^{\frac{3}{2}}},$$

consequently, this one, which is the same thing,

$$X = \frac{\pi}{16}\int\Big(1-\frac{u}{\sqrt{1+u^2}}\Big)du, \tag{4}$$

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[1](⇓) By substituting z^2 in the denominator of the expression (3) with $z^2 = y^2\Big[1+(u+u')^2\Big]$ and dz in the numerator of it,

$$\begin{aligned} X = &\ \frac{1}{4}\iiint \frac{dy\sqrt{1+(u+u')^2}dudu'}{\Big[1+(u+u')^2\Big]^{\frac{3}{2}}y^2} = \frac{1}{4}\iiint \frac{dydudu'}{\Big[1+(u+u')^2\Big]^{\frac{3}{2}}+y^2\Big[1+(u+u')^2\Big]^{(2-\frac{1}{2})}} \\ = &\ \frac{1}{4}\iiint \frac{dydudu'}{\Big[1+(u+u')^2\Big]^{\frac{3}{2}}\Big(1+y^2\Big)} = \frac{1}{4}\iint \frac{dudu'}{\Big[1+(u+u')^2\Big]^{\frac{3}{2}}}\underbrace{\int_0^\infty \frac{dy}{\Big(1+y^2\Big)^2}}_{\frac{\pi}{4}}. \end{aligned}$$

In respect to $\int_0^\infty \frac{dy}{\left(1+y^2\right)^2} = \frac{\pi}{4}$, we refer to [9, p.222] as $a=b=1, n=2$.

$$\int_0^\infty \frac{dx}{(ax^2+b)^n} = \frac{(2n-3)!!}{(2n-2)!!}\frac{\pi}{2b^n}\sqrt{\frac{b}{a}}, \quad a,b>0,$$

where, $(2n-3)!! = (2n-3)(2n-5)\cdots 5\cdot 3\cdot 1$, $\quad (2n)!! = 2n(2n-2)\cdots 4\cdot 2$, $\quad 0!! = 1$. .

[2](⇓) The integration of (4) is gotten as follows : we put at first $1+(u+u')^2 \equiv t \to 2(u+u')du' = dt$, then

$$\int\int \frac{dudu'}{[1+(u+u')^2]^{\frac{3}{2}}} = \int \frac{dt}{t^{\frac{3}{2}}} = \int_0^\infty \frac{2(u+u')du'}{2\big[1+(u+u')^2\big]^{\frac{1}{2}}} = \Big[\frac{u+u'}{[1+(u+u')^2]^{\frac{1}{2}}}\Big]_{u'=0}^{u'=\infty} = 1-\frac{u}{\sqrt{1+u^2}}.$$

namely, finally, [3]

$$X = \frac{\pi}{16}, \qquad q = \frac{\pi\rho^2}{8}\int_0^\infty r^4\varphi(r)dr. \tag{5}$$

§ 17. (The necessity to regard to the variation of the density of the liquid near the wall of the tube.)

$$(9)_3 \qquad Q + Q' + P = 0, \tag{6}$$

where, $Q = \Delta$, for the equilibrium of this part of the liquid.

The force Q' won't be differ sensibly from the force R of the (no. 14) ; because it would be between them in the ratio of the contour c of the base a to that of the base b, which we can take the one for the other. Therefore, we will have $Q' = R = cq$.

On the force P, its expression will differ from that of R in quintuplicate integral, with the sign of u' and with the limits relative to u and u', namely, that we will have

$$P = 2\rho^2 c\int\int\int\int\int \varphi(r)\frac{z+z'}{r}dzdz'dudu'dx, \qquad r^2 = x^2 + (u-u')^2 + (z+z')^2 \ ;$$

the integrals relative to x, z, z', being always zero and infinity ; however, those which responds to u and u' isn't extending only from zero to l, in designating with l the length of KL.

$$P = 2\rho^2 c\int\int\int\int\int \varphi(r')\frac{(u-u')u}{r'}dzdxdudu', \qquad (r')^2 = x^2 + z^2 + (u-u')^2.$$

Let again $x = y\cos\nu$, $z = y\sin\nu$. If we substitute these variables y and ν to x and z, it will need to take $dxdz = ydyd\nu$; the limits which respond to ($x = 0$ and $z = 0$) and ($x = \infty$ and $z = \infty$) will be ($y = 0$ and $\nu = 0$), ($y = \infty$ and $\nu = \frac{1}{2}\pi$) ; in effectuating the integration relative to ν, it will result then

$$P = -\pi\rho^2 c\int_0^\infty\int_0^l\int_0^l \varphi(r')\frac{(u-u')u}{r'}ydydudu', \qquad (r')^2 = y^2 + (u-u')^2.$$

Consequently, this triple integral is the same with that which exists in the expression of V of the (no. 8) ; in the analysis of the (no. 9), we will conclude then

$$P = -\frac{1}{4}\pi\rho^2 c\int_0^\infty r^4\varphi(r')dr = -2cq,$$

in neglecting always the term which would have the factor l, and regarding to the value of q of the (no. 16). These values of Q, Q', P, reduce the equation (6) to $\Delta = cq$.

Consequently, for that this value of Δ is accord with that which is given with the equation (2), it might need that it has been $q' = q$; this would cause that the material of the tube would has been the same with that of the liquid. QED.

§ 18. (The expression of Q in sextuple integral won't differ hence with that of R of the (no. 15).)

Instead to determine the force Q owing to the condition of the equilibrium of D, we can obtain from it the value with the direct integrations, in continuing to neglect the variation of

[3]($\Downarrow$) The integration of (4) is deduced as follows : if we put $\sqrt{1+u^2} \equiv t$, then $1 + u^2 = t^2, \rightarrow 2udu = 2tdt \rightarrow udu = tdt$.

$$\int \frac{u}{\sqrt{1+u^2}}du = \int \frac{tdt}{t} = \int dt = t + C = \sqrt{1+u^2} + C.$$

$$\int \Big(1 - \frac{u}{\sqrt{1+u^2}}\Big)du = \int du - \int \frac{u}{\sqrt{1+u^2}} = \Big[u\Big]_0^\infty - \Big[\sqrt{1+u^2}\Big]_0^\infty = \infty - (\infty - 1) = 1.$$

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the density of the liquid near its superior surface. The expression of Q in sextuple integral will differ hence with that of R of the (no. 15) only with the sign of z' and with the limits in respect to z and z', so that we will have

$$Q = \rho^2 \int\int\int\int\int\int \varphi(r)\frac{z+z'}{r}dzdz'dudu'dxds, \quad r^2 = x^2 + (u+u')^2 + (z-z')^2.$$

in conserving all the notations of the number cited, and integrating in respect to $z$and z', from the plane GF to its tangential plane. If we suppose that the plane of the figure, which is already vertical, were additionally perpendicular to the cylindrical wall of the tube, it is evident that the tangent plane at O will be perpendicular to the plane ; because it wouldn't have any reason for which it is inclined rather from one side than from the other. Posed thus, the values of z and z', which respond to the second limits of the integrals, will be independent of x and the form $z = y + \theta u, \qquad z' = y - \theta u'$, in designating with the ordinate OK of the point O, and with θ the tangent of the inclination of the plane at O on the horizontal plane passing with the same point. We will take for this inclination the angle $\angle K'ON$ composed between the prolongation OK' from OK and the exterior normal ON ; angle which we will regard as positive or as negative, according as ON will fall outward of inward of the two parallels $K'K$ and DL. It is followed that if we designate with ω the angle $\angle NOM$ which make the normal ON with the perpendicular OM fell from the point O on the straight DE, and which will be obtuse or acute, according as the curve AOB will turn at the point, it concavity or its upward convexity, we will have $\theta = -\cot\omega$. In naming Z the double integral to z and z', we will have then

$$Z = \int_0^{y+\theta u}\int_0^{y-\theta u} \varphi(r)\frac{z'-z}{r}dzdz'.$$

Let $\Phi(r)$ be the same function with in the (no. 9), so that we would have

$$\varphi(r) = -\frac{d\Phi(r)}{dr}, \qquad \varphi(r)\frac{z'-z}{r} = -\frac{d\Phi(r)}{dz'};$$

it will result then

$$Z = \int_0^{y+\theta u} \Phi(r')dz - \int_0^{\infty} \Phi(r_\imath)dz.$$

in supposing

$$(r')^2 = x^2 + (u+u')^2 + (y - \theta u' - z)^2, \quad r_\imath^2 = x^2 + (u+u')^2 + z^2,$$

and observing that we can replace the limit $z = y + \theta u$ with the infinity in the second integral. If we put, in the primary, $y - \theta u' - z = \zeta, \qquad dz = -d\zeta$, the limits relative to ζ will be $y - \theta u'$ and $-\theta(u+u')$. We will replace the primary limit with $\zeta = \infty$, by reason of that $y - \theta u'$ to a value sensible for all the values insensible of u', of which are the only one which doesn't make with r' sensible ; we will have then

$$\begin{aligned} Z &= -\int_\infty^{-\theta(u+u')} \Phi(r')d\zeta - \int_0^\infty \Phi(r_\imath)dz = -\int_0^{-\theta(u+u')} \Phi(r')d\zeta, \\ r^2 &= x^2 + (u+u')^2 + \zeta^2. \end{aligned}$$

in using the letter ζ instead of z, in the second integral, this one, which changes $r_\imath$ with r', and reducing next the two integrals to only one ; and if we put $-(u+u')\zeta$ and $-(u+u')d\zeta$ instead of ζ and $d\zeta$, we will conclude then

$$Z = \int_0^\theta r'\Phi(r')\frac{u+u'}{r'}d\zeta, \quad r^2 = x^2 + (u+u')^2(1+\zeta^2).$$

It will permit now to extend the integrals relative to u and u' from zero to infinity ; this one, which responds to x will have for limits $\pm\infty$, or only $x = 0$ and $x = \infty$, in doubling the result.

We will have then

$$\int\int\int Zdudu'dx = 2\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\theta r'\Phi(r')\frac{u+u'}{r'}dudu'dx.$$

Let be again, as in the (no. 9), $r'\Phi(r') = -\frac{d\Phi'(r')}{dr'}$; it will result then

$$r'\Phi(r')\frac{u+u'}{r'} = -\frac{1}{\zeta^2}\frac{d\Phi'(r')}{du'},$$

and in consequence,

$$\int\int\int Zdudu'dx = 2\int_0^\infty\int_0^\infty\int_0^\theta \Phi(r')\frac{dudxd\zeta}{1+\zeta^2}, \quad r^2 = x^2 + u^2(1+\zeta^2).$$

by reason that $\Phi'(r')$ evaporates at the limit $u' = \infty$. If hence we put xu and xdu instead of u and du, and if we use r instead of r', we will have

$$\int\int\int Zdudu'dx = 2\int_0^\infty\int_0^\infty\int_0^\theta \Phi(r)\frac{xdxdud\zeta}{1+\zeta^2}, \quad r^2 = x^2\left[1 + u^2(1+\zeta^2)\right].$$

this last equation turns into

$$xdx = \frac{rdr}{1+u^2(1+\zeta^2)} \ ;$$

from the above, we conclude

$$\int\int\int Zdudu'dx = 2\int_0^\infty \Phi(r')rdr\int_0^\infty\int_0^\infty \frac{dud\zeta}{(1+\zeta^2)\left[1+u^2(1+\zeta^2)\right]},$$

namely,

$$\int\int\int Zdudu'dx = \frac{\pi\theta}{\sqrt{1+\theta^2}}\int_0^\infty r\Phi'(r)dr,$$

in effectuating the integration relative to u, and next, that which responds to ζ.

Owing to this reduction of the integral in respect to z, z', u, u', x, and in putting for θ its value $-\cot\omega$, the expression of Q will turn into

$$Q = -\pi\rho^2\int_0^\infty r\Phi'(r)dr. \int \cos\omega ds.$$

In integrating by parts, it turns into

$$\int_0^\infty r\Phi'(r)dr = \frac{1}{2}\int_0^\infty r^3\Phi(r)dr = \frac{1}{8}\int_0^\infty r^4\varphi(r)dr \ ;$$

from the above, we conclude (no. 16)

$$(10)_3 \qquad Q = -q\int \cos\omega ds. \tag{7}$$

This integral relative to ds will turn to be extended at the entire contour from the base of D. Consequently, I say that there will need to consider $\cos\omega$ as the quantity constant. In effect, if we trace with the extremities of ds two planes perpendiculars to this element, $-q\cos\omega ds$ will be the part of Q which will activate on the surface of C composed between these two planes ; the action vertical of the excess of C on this segment will be composed of forces which is destroyed two by two ; we will be able to neglect the weight of this segment with ratio to the force $-q\cos\omega ds$, which will turn into make equilibrium at the part of the forces Q' and P which activate on this same segment, of which forces isn't vary in passing of the point K to another point of the contour of D ; it will need hence that $\cos\omega$ doesn't vary no more ; and that keeps to this one, which the quantities depend on the curvature of the tube, which change from a point to another when the base isn't circular, disappear from the expression of the forces which we consider, hence, we have seen in the (no. 15), at the regard of another force R of the same nature. In naming hence c the contour entire of the base of D, we will have finally $Q = -cq\cos\omega$, for the value of Q which would need to calculate.

2. Application of Legendre's elliptic functions to the capillary action.

We introduce Poisson's applications of the primary and secondary kinds of Legendre's elliptic function to the capillary action in § 86 – 89 and § 91 – 92 of [10]. He also applies third kind of the elliptic function to the calculus of heat theory, in § 215 – 216 of [11], however we omit this kind for lack of space.

§ 86.

Let discuss again the question of the (no. 57), relative to the equilibrium of a liquid contained between two planes vertical and parallel, which the one will regard as indefinitely prolonged in the horizontal sens, in order not to have to consider this one, which arrives to their extremities.

If we take the plane of the x and y and z parallel to two plane given, the ordinate z will be independent of y, and the equation (8) [4] of the (no. 49) will be reduced to

$$\frac{a^2\frac{d^2z}{dx^2}}{\left(1+\frac{d^2z}{dx^2}\right)^{\frac{3}{2}}} = 2z \tag{9}$$

[5] in putting, as previously, $H = g\rho a^2$. In multiplying with $-dz$, integrating and designating with b an arbitrary constant, we will have

$$(1)_6 \qquad \frac{a^2}{\left(1+\frac{d^2z}{dx^2}\right)^{\frac{1}{2}}} = b - z^2. \tag{10}$$

This equation will be that of a section of the surface of liquid, made with a plane vertical and perpendicular at the two given planes. The z positive are regarded in sense contrary of the gravity, and from the level exterior of liquid in which the two planes are prolonged with their extremities inferior. The radical needs to be positive or negative, according as the normal to the curve makes an angle acute or obtuse with the direction of the z positive ; however, in all its length, the curve is met only at the only point with each vertical, and the angle which it activates is always acute, so that the radical will be constantly positive.

At the two extremities of the layer, the cosines of the angle composed between the normal and a horizontal, take each other outward of liquid, is a given quantity, which can be positive or negative.

- If it is positive in these two points, the curve will turn its convexity with upward, in all its length ;
- if it is negative the curve will be concave upward ;

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[4]($\Downarrow$) The equation (8) is as follows :

$$(1)_4 \qquad g\rho z = \frac{1}{2}H\left(\frac{1}{\lambda}+\frac{1}{\lambda'}\right)\ ; \tag{8}$$

[5]($\Downarrow$) cf. The left hand-side of expression (9) is as follows :

$$\frac{a^2\frac{d^2z}{dx^2}}{\left(1+(\frac{dz}{dx})^2\right)^{\frac{3}{2}}} = \int\frac{axdx}{\sqrt{\left(a^2+x^2\right)^3}}.$$

- if it is positive to the one of extremities and negative to the other, the curve will be composed, in general, of two parts which turn their concavity into contrary sens.

I say *in general* ; because I will investigate that there is some states of the equilibrium possible, for which this last case returns in the one of two primaries.

- In these primary cases, there will have one point C for which the tangent will be horizontal,
- and in the last, the curve will present generally a point of inflection which I will call I.

I am going to consider successively the case of the point C and that of point I, which are essentially distinct.

§ 87. (I am going to consider successively the case of the point C and that of point I, which is essentially distinct.)

I designate with h the value of z which responds to the point C ; by reason of $\frac{dz}{dx} = 0$ in this point, we will have $a^2 = b - h^2$; and in eliminating b, the equation (10) will turn into

$$(2)_6 \qquad \frac{a^2}{\left(1+\left(\frac{dz}{dx}\right)^2\right)^{\frac{1}{2}}} = a^2 + h^2 - z^2. \tag{11}$$

The radical being a positive quantity, it needs that z^2 weren't greater than $a^2 + h^2$; and, by reason of that the left hand-side of the equation is less than a^2, it needs that z^2 were not smaller than h^2. Then, we see already that without consideration of the sign, the variable z is composed between the limits h and $\sqrt{a^2 + h^2}$; it will be positive or negative, according as the curve will turn its concavity or its convexity with upward.

We get from this equation

$$dx = \frac{(a^2 + h^2 - z^2)dz}{\sqrt{(z^2 - h^2)\,(h^2 + 2a^2 - z^2)}}.$$

I will consider separately the two parts of the curve which arrives at the point C ; in each of them, the variable x will be regarded as positive and regarded from this point ; and for that it cross from this point up to each edge of the curve, I will suppose the radical of the same sign with dz.

Posed thus, to explain x in elliptic function, I put $z^2 = \frac{(h^2+2a^2)h^2}{h^2+2a^2\cos\varphi}$; from here we get

$$\tan^2\varphi = \frac{(h^2 + 2a^2)(z^2 - h^2)}{h^2(h^2 + 2a^2 - z^2)};$$

and the variable z^2 is neither less than h^2, nor greater than $h^2 + a^2$, this value of $\tan^2\varphi$ will be positive ; this one suffices for that φ were a real angle. The expression of dx will turn

$$dx = \frac{(a^2 + h^2)d\varphi}{\sqrt{h^2 + 2a^2\cos^2\varphi}} - \frac{\left(h^2 + 2a^2\right)h^2 d\varphi}{\left(h^2 + a^2\cos^2\varphi\right)^{\frac{3}{2}}},$$

consequently, this one is the same thing,

$$dx = \frac{(2 - c^2)a}{c\sqrt{2}}\,\frac{d\varphi}{\sqrt{1 - c^2\sin^2\varphi}} - \frac{2(1 - c^2)a}{c\sqrt{2}}\,\frac{d\varphi}{(1 - c^2\sin^2\varphi)^{\frac{3}{2}}},$$

in designating with c a quantity positive, less than the unit, and given with the equation $c^2 = \frac{2a^2}{2a^2+h^2}$.

Additionally, we have identically

$$d\Big(\frac{\sin\varphi\cos\varphi}{\sqrt{1-c^2\sin^2\varphi}}\Big)=\frac{1}{c^2}\sqrt{1-c^2\sin^2\varphi}d\varphi-\frac{(1-c^2)}{c^2}\,\frac{d\varphi}{(1-c^2\sin^2\varphi)^{\frac{3}{2}}};$$

from here, it results

$$dx=\frac{(2-c^2)a}{c\sqrt{2}}\,\frac{d\varphi}{\sqrt{1-c^2\sin^2\varphi}}-\frac{2a}{c\sqrt{2}}\,\sqrt{1-c^2\sin^2\varphi}d\varphi+ac\sqrt{2}d\Big(\frac{\sin\varphi\cos\varphi}{\sqrt{1-c^2\sin^2\varphi}}\Big)$$

Owing to the notation known of Mr. Legendre, we have also

$$\int\frac{d\varphi}{\sqrt{1-c^2\sin^2\varphi}}=F(c,\varphi),\qquad\int\sqrt{1-c^2\sin^2\varphi}d\varphi=E(c,\varphi)\ ;$$

the integrals starting with the variable φ. In integrating, we will have then

$$(3)_6\qquad\frac{x\sqrt{2}}{a}=\frac{2-c^2}{c}F(c,\varphi)-\frac{2}{c}E(c,\varphi)+\frac{c\sin 2\varphi}{\sqrt{1-c^2\sin^2\varphi}}\tag{12}$$

[6] We don't add the constant arbitrary, because that x is null at the point C, for which we have $z=h$, this one, which gives $\varphi=0$ and makes evaporate the right hand-side of this equation. We will have at the same time

$$(4)_6\qquad z^2=\frac{2a^2(1-c^2)}{\sqrt{1-c^2\sin^2\varphi}}\tag{13}$$

and these equations (12) and (13) make known the x and z of each of the points of the curve, the functions of a third variable φ, when we will have determined the module c.

Consequently, if we put $\frac{\frac{dz}{dx}}{\left(1+\left(\frac{dz}{dx}\right)^2\right)^{\frac{1}{2}}}=-\cos\omega$, ω will be the angle which is given at the two extremities of the layer, and which depend, at each of these points,

- on the material of corps terminated with the vertical plane,
- and on that of liquid.

In designating with k the value of z which responds to the one of these two points, and eliminating $\frac{dz}{dx}$ in the equation (11) and the precedent, it turns into $k=h^2+a^2(1-\sin\ \omega)$; in regarding to the value of c^2, we will have then

$$(5)_6\qquad h^2=\frac{2a^2(1-c^2)}{c^2},\qquad k^2=\frac{a^2}{c^2}(2-c^2-c^2 sin\ \omega)\ ;\tag{14}$$

and if we call θ the value of φ which responds to $z=k$, it will result

$$(6)_6\qquad\tan^2\theta=\frac{1-\sin\omega}{(1+\sin\omega)(1-c^2)}.\tag{15}$$

[6]($\Downarrow$) We call it Legendre-Jacobi's standard expression :

$$\begin{aligned}
\text{The primary elliptic integral : } F(k,\varphi) &= \int_0^{\varphi}\frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}}=\int_0^{\sin\varphi}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},\\
\text{The secondary elliptic integral : } E(k,\varphi) &= \int_0^{\varphi}\sqrt{1-k^2\sin^2\varphi}d\varphi=\int_0^{\sin\varphi}\sqrt{\frac{1-k^2t^2}{1-t^2}}dt,\\
\text{The tertiary elliptic integral : } \mathit{\Pi}(\varphi,n,k) &= \int_0^{\varphi}\frac{d\varphi}{(1+n\sin^2\varphi)\sqrt{1-k^2\sin^2\varphi}}\\
&= \int_0^{\sin\varphi}\frac{d\varphi}{(1+nt^2)\sqrt{(1+nt^2)(1-k^2t^2)}}.
\end{aligned}$$

where k the parameter, and when $\varphi=\frac{1}{2}\pi$, we call the top two are complete elliptic integrals. [8, p.1731].

Let α be the value corresponding to x, namely, the distance from the point C to the one of two vertical planes ; we will have

$$(7)_6 \qquad \frac{x\sqrt{2}}{a} = \frac{2-c^2}{c}F(c,\theta) - \frac{2}{c}E(c,\theta) + \frac{c\sin 2\theta}{\sqrt{1-c^2\sin^2\theta}} \qquad (16)$$

If we designate with α' and ω' the distance and the angle relative to the other vertical plane, and with θ' this one turns into θ, when we put ω' instead of ω, we will have a second equation which will be deduced from the precedent, in changing α and θ, with α' and θ'. I add these two equations, and I put $\alpha + \alpha' = \delta$, so that δ were the distance composed in the two vertical planes ; it turns

$$(8)_6 \qquad \begin{aligned} \frac{x\sqrt{2}}{a} &= \frac{2-c^2}{c}\Big[F(c,\theta)+F(c,\theta')\Big] - \frac{2}{c}\Big[E(c,\theta)+E(c,\theta')\Big] \\ &+ \frac{c\sin 2\theta}{\sqrt{1-c^2\sin^2\theta}} + \frac{c\sin 2\theta'}{\sqrt{1-c^2\sin^2\theta'}} \end{aligned} \qquad (17)$$

for the equation which will serve to determine c.

In the case which we examine, the angles ω and ω' will be all the two acute or all the two obtuse ; according as they will be obtuse or acute, the curve will be concave or convex upward, and we will take with the sign $+$ or with $-$, the values of z and h given with the equation (13) and the primary equation (14). In calling k', this one, which turns k when we change ω with ω', the extremities ordinates k and k' are deduced from the formula (13), in putting $\varphi = \theta$ and $\varphi = \theta'$, namely they will be given immediately with the second equation (14). The distance from C to the one of the vertical planes or the one of the extreme values of x, will be determined by means of the formula (16) ; and in subtracting from δ, we will have the distance from C' to the other plane.

§ 88. (An example for calculation using the elliptic integral by Legendre's tables.)

When we will have $\omega = \omega'$, the distances α and α' will be equal between them and to $\frac{1}{2}\delta$. If these angles are, in additionally, zero or π, we will have simply $\cos\theta = \sqrt{1-c^2}$.

To consider with relation to $c$the equation (16) which is transcendental, it would need to give to c a series of values ascending with the very small differences, from $c = 0$ to $c = 1$; let calculate by means of elliptic tables by Mr. Legendre, the value corresponding to the right hand-side of this equation[a] ; and let form then a table of the values of $\frac{\alpha\sqrt{2}}{a}$, relative to all these value of c : these being, when the distance δ or $s\alpha$, and constant a, and in consequence, the quantity $\frac{\alpha\sqrt{2}}{a}$ would be given, we would seek in this table, the value corresponding to c.

[a] (⇓) cf. This means the equation (16).

But, the problem will be moreover simple

- if we give the elevation h of this point C and the constant a,
- and if we demand how long it must exist that the distance 2α composed between two planes.

Let suppose, for example, which we musty have $h^2 = 2a^2$; it will result $c = \frac{1}{\sqrt{2}}$, $\cot\theta = \frac{1}{\sqrt{2}}$, $\theta = 54°44'$, and the equation (16) will turn

$$\frac{\delta}{h} = \frac{3}{\sqrt{2}}F(c,\theta) - 2\sqrt{2}E(c,\theta) + \sqrt{\frac{2}{3}}.$$

For these values of c and θ, the tables by Mr. Legendre give

$$F(c,\theta) = 1.02806, \quad E(c,\theta) = 0.89111 \ ;$$

from here, we conclude $\frac{\delta}{h} = 0.4776$, for the ratio of the distance from the two planes to the smallest ordinate of the curve.

Owing to the equation (14), the greatest ordinate is $k = h\sqrt{\frac{3}{2}}$; the average ordinate is then $z = \frac{1}{2}h\left(1 + h\sqrt{\frac{3}{2}}\right)$; and the value corresponding to φ will be $\varphi = 38°16'30''$. For this value of φ and $c = \sin 45°$, we see, in the tables of Mr. Legendre,

$$F(c,\theta) = 0.69500, \quad E(c,\theta) = 0.64437 \ ;$$

and the equation (12) gives successively $x = h(0.2061)$. In comparing this value of x with that of α, consequently of $\frac{1}{2}\delta\ x = \alpha(0.8628)$; so that, in this example, the average ordinates are very more approached from the planes vertical than the point C ; this one give at the few of curvature of the liquid near this point.

§ 89.

If h is very great with relation to a, the module c will be a very small function, and the one will be capable to develop the elliptic functions in very convergent series, following the power of c ; this one, which gives

$$F(c,\theta) = \theta + \frac{c^2}{4}(\theta - \sin\theta\cos\theta) + \text{etc}, \quad E(c,\theta) = \theta - \frac{c^2}{4}(\theta - \sin\theta\cos\theta) + etc.$$

The equation (16) will turn into

$$\frac{x\sqrt{2}}{a} = c\sin\theta\cos\theta + \frac{c^2}{8}(\theta - \sin\theta\cos\theta + 6\sin^3\theta\cos\theta),$$

in neglecting the fifth power of c. For more simply, I will suppose that ω were zero or π ; we will have then, in virtue of the equation (15),

$$\tan\theta = 1 + \frac{1}{2}c^2 + \text{etc.}, \qquad \theta = \frac{1}{4}\pi + \frac{1}{4}c^2 + \text{etc.},$$

and in consequence,

$$\frac{\alpha\sqrt{2}}{a} = \frac{1}{2}c + \frac{c^2}{8}\left(1 + \frac{1}{4\pi}\right).$$

It will need that the ratio were a very small function, which we will neglect the fifth power ; from here, it will result

$$c = \frac{2\alpha\sqrt{2}}{a} - \frac{4\alpha^3\sqrt{2}}{a^3}\left(1 + \frac{1}{4\pi}\right).$$

The primary equation (14) gives successively

$$h = \pm\left[\frac{a^2}{2\alpha} - a\left(1 + \frac{1}{4\pi}\right)\right], \tag{18}$$

where, we will take the sign superior or inferior, according as ω will be zero of π, namely according as the liquid will be concave or convex upward.

If the two vertical planes which terminate the liquid are from the same nature, namely if ω' is also zero or π, so that 2α were their mutual distance, we see that the primary term of this value of h will be the same with in the case of a tube which would have this distance for radius ; this one, which doesn't hold for the total value of h.

Mr. Gay-Lussac has observed the elevation of the water between two planes vertical and parallel, previously immersed with this liquid. Their distance being $2\alpha = 1.069^{mm}$, he has found $h = 13.574^{mm}$. However, the temperature is elevating to 16° during this experiment, it needs to augment the value of h, to make the value of a^2, which we will deduce, comparable to that of (no. 56), which responds to a temperature of 8.5°. We will take then

$$h = \left(13\ ^{mm}.574\right)\left(1 + \frac{7.5}{2300}\right) = 13.5872,$$

in reason that the augmentation of density of the water is about $\frac{1}{2300}$ for each degree of fall in the temperature, and that the increment of h is proportional to this augmentation (no. 53). By means of this last value of h and of that of 2α, we see $a^2 = 14.6473$ square millimeters ; this one, which differs few from the value of a^2 of (no. 56), which it will need to prefer, as being concluded from an observation more susceptible of exactness.

§ 91.

I pick up the equations (12) and (16) each other, and I put $\alpha - x = u$, so that u were the distance of a point certain from the curve of liquid at the vertical plane which responds to the angle ω ; it turns into

$$\begin{aligned}\frac{u\sqrt{2}}{a} &= \frac{2-c^2}{c}\Big[F(c,\theta) - F(c,\varphi)\Big] - \frac{2}{c}\Big[E(c,\theta) - E(c,\varphi)\Big] \\ &+ \frac{c\sin 2\theta}{\sqrt{1-c^2\sin^2\theta}} - \frac{c\sin 2\varphi}{\sqrt{1-c^2\sin^2\varphi}}\end{aligned}$$

Now, if we suppress the another vertical plane, the curve will be asymptotic of the axis of u, and the ordinate h of the point C will be infinitely small ; in calling hence b the compliment of module, so that we would have $1-c^2 = b^2$, b^2 will be also infinitely small, in virtue of the primary equation (14) ; and if we put, to abridge, $\frac{1+\sin\omega}{1-\sin\omega} \equiv (\theta')^2$, the equation (15) will give $\theta = \frac{1}{2}\pi - \theta' b$. Owing to the equation (13), z will be infinitely small, except for the values of φ infinitely few different from $\frac{1}{2}\pi$; I put hence also $\varphi = \frac{1}{2}\pi - \varphi' b$; and it results $z^2 = \frac{2a^2}{1+(\varphi')^2}$, namely, $\varphi' = \frac{1}{z}\sqrt{2a^2 - z^2}$. For these values of φ and θ, we will have $E(c,\theta) - E(c,\varphi) = \int_\varphi^\theta \sqrt{1-c^2\sin^2 v}dv = 0$. We have also $F(c,\theta) - F(c,\varphi) = \int_\varphi^\theta \frac{dv}{\sqrt{1-c^2\sin^2 v}}$; and in putting $v = \frac{1}{2}\pi - bv'$, $dv = -bdv'$, this last equation turns into

$$F(c,\theta) - F(c,\varphi) = \int_{\varphi'}^{\theta'} \frac{dv'}{\sqrt{1+(v')^2}} = \log\frac{\varphi' + \sqrt{1+(\varphi')^2}}{\theta' + \sqrt{1+(\theta')^2}}. \qquad (19)$$

[7] We have additionally

$$\frac{c\sin 2\theta}{\sqrt{1-c^2\sin^2\theta}} = \frac{2\theta'}{\sqrt{1+(\theta')^2}}, \qquad \frac{c\sin 2\varphi}{\sqrt{1-c^2\sin^2\varphi}} = \frac{2\varphi'}{\sqrt{1+(\varphi')^2}}\ ;$$

$$\frac{\alpha\sqrt{2}}{a} = \log\frac{\varphi' + \sqrt{1+(\varphi')^2}}{\theta' + \sqrt{1+(\theta')^2}} + \frac{2\theta'}{\sqrt{1+(\theta')^2}} - \frac{2\varphi'}{\sqrt{1+(\varphi')^2}}\ ;$$

[7](⇓) The equation (19) turns into

$$\int_{\varphi'}^{\theta'} \frac{dv'}{\sqrt{1+(v')^2}} = \left[\log\left(x + \sqrt{1+x^2}\right)\right]_{\varphi'}^{\theta'} = \log\frac{\varphi' + \sqrt{1+(\varphi')^2}}{\theta' + \sqrt{1+(\theta')^2}}.$$

and if we substitute for φ' and θ' their values, and that we put $\omega = \frac{1}{2}\pi + \mu$, it will finally

$$(9)_6 \qquad \frac{\alpha\sqrt{2}}{a} = \log \frac{\left(a\sqrt{2} + \sqrt{2a^2 - z^2}\right)\sin\frac{1}{2}\mu}{z\left(1 + \cos\frac{1}{2}\mu\right)} + 2\cos\frac{1}{2}u - \frac{2\sqrt{2a^2 - z^2}}{a\sqrt{2}}, \qquad (20)$$

for the equation of the curve formed with the liquid which is elevated or fell along a vertical plane ; μ being the acute angle which make the normal to its extremity, with the vertical, and which we will regard as positive or as negative, according as this curve will turn its concavity or its concavity upwards, in order to that the angle ω were obtuse in the first case and acute in the second. The variable z will be, consequently, of the same sign with μ and the radical $\sqrt{2a^2 - z^2}$ will turn into be positive, for that we would have $u = \infty$, when $z = 0$, hence that we have supposed so.

If we call l the value of z which responds to $u = 0$, we will have, owing to the second equation (14), $l = a\sqrt{2}\sin\frac{1}{2}\mu$; and this quantity makes effectively zero the formula (20), when we substitute it with z. It results that the constant a, relative to the material of a liquid, will explain its elevation over the level, along a vertical plane which have been previously immersed in all its height with this same liquid ; because we have then $\omega = \pi$, $\mu = \frac{1}{2}\pi$, and consequently, $l = a$. In the case of the water at the temperature of 8.5 ° (no. 56), it will turn into $l = a = 3.8888$ ^{mm}, for this elevation.

§ 92.

Now, let consider the case where the curve of the liquid presents a point of inflection I.

I will designate with i the angle unknown that make the tangent at this point with the horizontal straight, traced with the same point. As it needs that the part of the curve which turn its concavity through in height, were more elevated than the concave part through in below, it follows that the angle i will be smaller than every of the acute angles which make the tangents at the extreme points, with the horizontal straight traced with the point I, and smaller than a right angle when these tangent will be vertical. Moreover, $\frac{d^2z}{dx^2}$ changing of sign of part and of the other of I, it will need that this quantity were zero or infinite at this point ; the ordinate z will be, at the same time, zero or infinite, in virtue of the primary equation of the (no. 86) ; and because it would know to be infinite, it will need that the quantities $\frac{d^2}{dx^2}$ and z were null at the point I, which will find, in consequence, in the plane of the level of the liquid.

I make hence $z = 0$ and $\frac{dz}{dx} = \tan i$ in the equation (10) ; it result $b = a^2\cos i$. This equation turns

$$\frac{a^2}{\left(1 + (\frac{dz}{dx})^2\right)^{\frac{1}{2}}} = a^2\cos i - z^2 \ ;$$

by reason of that the radical is a positive quantity, we see that z^2 will be smaller than $a^2\cos i$. We get from this

$$dx = \frac{\left(a^2\cos i - z^2\right)dz}{\sqrt{\left[z^2 + a^2(1 - \cos i)\right]\left[a^2(1 + \cos i) - z^2\right]}}.$$

I will fix at the point I the origin of x ; I will consider separately every of two parts of the curve which arrive at this point, and I will regard the radical as being of same sign with dz.

To explain x in elliptic functions, I put $\cos\frac{1}{2}i = c$, and next

$$(10)_6 \qquad z^2 = \frac{2a^2c^2(1 - c^2)\sin^2\varphi}{1 - c^2\sin^2\varphi} \qquad (21)$$

from the above, it results

$$\tan^2\varphi = \frac{z^2}{\Big(2a^2c^2 - z^2\Big)\Big(1 - c^2\Big)} \ ;$$

quantity positive, by reason of $z^2 < a^2 \cos i$; this suffices for that the angle were real. We will have

$$dx = \frac{ad\varphi}{\sqrt{2}\sqrt{1 - c^2\sin^2\varphi}} - \frac{(1-c^2)a\sqrt{2}d\varphi}{\Big(1 - c^2\sin^2\varphi\Big)^{\frac{3}{2}}},$$

or, this which is the same

$$dx = \frac{ad\varphi}{\sqrt{2}\sqrt{1 - c^2\sin^2\varphi}} - a\sqrt{2}(1-c^2)d\varphi d\varphi + c^2a\sqrt{2}d\Big(\frac{\sin\varphi\cos\varphi}{\sqrt{1 - c^2\sin^2\varphi}}\Big).$$

In integrating, we will hence

$$(11)_6 \qquad \frac{\alpha\sqrt{2}}{a} = F(c,\varphi) - 2E(c,\varphi) + \frac{2c^2\sin\varphi\cos\varphi}{\sqrt{1 - c^2\sin^2\varphi}} \qquad (22)$$

We don't add the arbitrary constant, because at the point I, we have, at once, $x = 0$, $z = 0$, $\varphi = 0$.

I designate with ω the same angle with previously (no. 87), with α the distance from the point I to the vertical plane, which responds to this angle, and with k the value of z relative to $x = \alpha$; so that we would have

$$\frac{\frac{dz}{dx}}{\Big(1 + \big(\frac{dz}{dx}\big)^2\Big)^{\frac{1}{2}}} = -\cos\omega, \qquad \frac{a^2}{\Big(1 + \big(\frac{dz}{dx}\big)^2\Big)^{\frac{1}{2}}} = a^2\cos i - k^2,$$

for $x = \alpha$. In eliminating $\frac{dz}{dx}$, it turns into

$$(12)_6 \qquad k^2 = a^2\Big(\cos i - \sin\omega\Big). \qquad (23)$$

The integral ω will be obtained for the vertical plane, along which the liquid is elevated, and acute for the another plane, along which it is fell ; the sign of k will be contrary to that of $\cos\omega$. In calling θ the angle φ which responds to $z = k$ and $x = \alpha$, we will have

$$\tan^2\theta = \frac{2c^2 - 1 - \sin\omega}{\Big(1 - c^2\Big)\Big(1 + \sin\omega\Big)} \ ; \qquad (24)$$

and, in virtue of the equation (22),

$$(13)_6 \qquad \frac{\alpha\sqrt{2}}{a} = F(c,\theta) - 2E(c,\theta) + \frac{2c^2\sin\theta\cos\theta}{\sqrt{1 - c^2\sin^2\theta}} \qquad (25)$$

This equation will respond to the one of the extremities of the curve of the liquid. If we designate with α', ω', θ', this one, which turn into α, ω, θ, relatively, to the other extremity, we will have a second equation which will be deduced from that, in putting α' and θ' instead of α and θ ; in making the sum of these equations, and call δ the distance of two vertical planes which terminate the curve, we will have

$$(14)_6 \qquad \begin{aligned} \frac{\delta\sqrt{2}}{a} &= F(c,\theta) + F(c,\theta') - 2E(c,\theta) - 2E(c,\theta') \\ &+ \frac{2c^2\sin\theta\cos\theta}{\sqrt{1 - c^2\sin^2\theta}} + \frac{2c^2\sin\theta'\cos\theta'}{\sqrt{1 - c^2\sin^2\theta'}} \end{aligned} \qquad (26)$$

This last equation will serve to determine the module c, and in consequence, the angle i ; the equations (21) and (22) will give next the values of the coordinates z and x in function of a third variable φ ; this is one which is the complete solution of the problem.

When the angle i, under which the curve turn to cut the natural level of the liquid, will give, the precedent equation will make know immediately, by means of tables of the elliptic functions, the distance of the two planes which will turn to hold. We are going to make successively, on this angle, different suppositions.

3. Conclusions

The formulae deduced by Laplace and Gauss are identical, Poisson uses as a commonly known formula. Poisson emphasizes the variation of density in the neighbor of wall and surface, by which the fall or elevation occurred. Today's common knowledge teaches it to us by means of the surface tension, of which Poisson doesn't tell at all, however, the difference between capillarity and surface tension is vague. For example, capillary wave means the wave of surface tension. We can replace a part of action which Poisson uses with surface tension. Legendre is, we think, the only person in Poisson's all life, whom Poisson defeated in such academic arena in high esteem for the tremendous works by Legendre. Without his works, as Poisson says, his applications to the ellipsoid haven't put into practice.

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